Vector Spaces

IDEA: Abstract our understanding of linear systems...
↳ Build a language to prove more powerful theorems.

Def^n: A <u>(real) vector space</u> is a set V (whose elements are <u>vectors</u>) with operations

(hidden) closure axioms

$$+ : \underline{V} \times \underline{V} \longrightarrow \underline{V} \qquad (\text{vector addition})$$

$$\cdot : \mathbb{R} \times \underline{V} \longrightarrow \underline{V} \qquad (\text{scalar multiplication})$$

satisfying the following axioms:

① $u + v = v + u$ for all $u, v \in V$ (Commutativity of Addition)

② $u + (v + w) = (u + v) + w$ for all $u, v, w \in V$ (Associativity of Addition)

③ There is a vector $0 \in V$ such that for all $v \in V$ $0 + v = v$. (Zero vector)
NB: 0 is the <u>zero-vector</u>

④ For all $v \in V$ there is a vector $w \in V$ such that $v + w = 0$. (Additive inverses)
NB: usually we denote $w = -v$.

⑤ $a \cdot (u + v) = (a \cdot u) + (a \cdot v)$ for all $a \in \mathbb{R}$ and all $u, v \in V$. (Scalar distribution over vector addition)

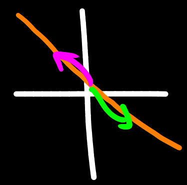
⑥ $(a + b) \cdot v = (a \cdot v) + (b \cdot v)$ for all $a, b \in \mathbb{R}$ and all $v \in V$. (Vector distribution over scalar addition)

⑦ $a \cdot (b \cdot v) = (ab) \cdot v$ for all $a, b \in \mathbb{R}$ and all $v \in V$. (Associativity of Scalar multiplication)

⑧ $1 \cdot v = v$ for all $v \in V$. (Scalar Identity).

Ex: $\mathbb{R}^n$ is a vector space for all n.
(we verified this awhile back).

Ex: Let $V = \{(x,y) \in \mathbb{R}^2 : x = -y\}$. With operations



$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

and $$c \cdot (x,y) = (cx, cy),$$

this set V is a vector space.

pf: First we need to show for $u, v \in V$ and $c \in \mathbb{R}$ we have $u + v \in V$ and $c \cdot v \in V$.
(i.e. closure of V under addition and scalar mult).

Let $u, v \in V$ and $c \in \mathbb{R}$. So $u = (u_1, u_2)$ and $v = (v_1, v_2)$ satisfy $\underline{u_1 = -u_2}$ and $\underline{v_1 = -v_2}$.

Now $u + v = (u_1, u_2) + (v_1, v_2) = (\underline{u_1 + v_1}, \underline{u_2 + v_2})$ and we know $\underline{u_1 + v_1} = \underline{(-u_2) + (-v_2)} = -(u_2 + v_2)$, so $u + v \in V$. On the other hand,

$cu = c(u_1, u_2) = (cu_1, cu_2)$ and because $u_1 = -u_2$, we have $cu_1 = c(-u_2) = -(cu_2)$, and hence $cu \in V$. Hence V is closed under vector addition and scalar multiplication.

Next we verify the 8 (remaining) conditions on a vector space:

Let $u = (u_1, u_2)$, $v = (v_1, v_2)$, $w = (w_1, w_2) \in V$ and $a, b \in \mathbb{R}$:

① (Commutativity):

$$u+v=(u_1,u_2)+(v_1,v_2)=(u_1+v_1,u_2+v_2)$$

$$=(v_1+u_1,v_2+u_2)=(v_1,v_2)+(u_1,u_2)=v+u \checkmark$$

② (Associativity of Vec. add.)

$$u+(v+w)=(u_1,u_2)+((v_1,v_2)+(w_1,w_2))$$

$$=(u_1,u_2)+(v_1+w_1,v_2+w_2)$$

$$=(u_1+(v_1+w_1),u_2+(v_2+w_2))$$

$$=((u_1+v_1)+w_1,(u_2+v_2)+w_2)$$

$$=(u_1+v_1,u_2+v_2)+(w_1,w_2)$$

$$=((u_1,u_2)+(v_1,v_2))+(w_1,w_2)$$

$$=(u+v)+w$$

③ We claim $0_v=(0,0)$ is the zero-vector for V.

Indeed, $0_v+v=(0,0)+(v_1,v_2)=(0+v_1,0+v_2)=(v_1,v_2)=v$.

Moreover, $0=-0$, so $(0,0)\in V$.

④ (Additive Inverses). For vector v, we have

$$v+(-v_1,-v_2)=(v_1,v_2)+(-v_1,-v_2)=(v_1-v_1,v_2-v_2)=(0,0)$$

on the other hand $(-v_1,-v_2)=-1\cdot(v_1,v_2)=-1\cdot v\in V$.

⑤ (distribution 1).

$$a\cdot(u+v)=a\cdot(u_1+v_1,u_2+v_2)=(a(u_1+v_1),a(u_2+v_2))$$

$$=(au_1+av_1,au_2+av_2)=(au_1,au_2)+(av_1,av_2)$$

$$=(a\cdot u)+(a\cdot v)$$

⑥ (distribution 2).

$$(a+b)\cdot v = ((a+b)v_1, (a+b)v_2)$$
$$= (av_1 + bv_1, av_2 + bv_2)$$
$$= (av_1, av_2) + (bv_1, bv_2)$$
$$= (a\cdot v) + (b\cdot v)$$

⑦ (Scalar association)

$$a\cdot(b\cdot v) = a\cdot(bv_1, bv_2) = (a(bv_1), a(bv_2))$$
$$= ((ab)v_1, (ab)v_2) = (ab)\cdot v$$

⑧ (Scalar Unit)

$$1\cdot v = 1\cdot(v_1, v_2) = (1v_1, 1v_2) = (v_1, v_2) = v$$

Hence V is a vector space under these operations! $\boxtimes$

Remark: These checks are mostly just the same work we did showing properties of vect. add. earlier...

Ex: Let $P_n(\mathbb{R})$ denote the set of polynomial's with real coefficients and degree at most n.

Let $+: P_n(\mathbb{R}) \times P_n(\mathbb{R}) \to P_n(\mathbb{R})$ be the usual polynomial addition, and scalar multiplication

$\cdot : \mathbb{R} \times P_n(\mathbb{R}) \to P_n(\mathbb{R})$ be the usual multiplication.

Then $P_n(\mathbb{R})$ is a vector space.

Special Case: When $n=3$, we have

$$\mathcal{P}_3(\mathbb{R}) = \{p(x) : p(x) \text{ has degree at most } 3\}$$
$$= \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 : a_0, a_1, a_2, a_3 \in \mathbb{R}\}$$

And the addition acts like so:

$$(a_0 + a_1 x + a_2 x^2 + a_3 x^3) + (b_0 + b_1 x + b_2 x^2 + b_3 x^3)$$
$$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3$$

and scalar multiplication works like that:

$$c(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = (ca_0) + (ca_1)x + (ca_2)x^2 + (ca_3)x^3$$

↳ Check the conditions are satisfied! $\boxtimes$

Ex: Let $m, n \geq 1$. The set

$$\mathcal{M}_{m,n}(\mathbb{R}) = \{A : A \text{ is an } m \times n \text{ matrix w/ real entries}\}$$

is a vector space under matrix addition and entry-wise scalar multiplication. $\boxtimes$

Ex: Let $V = \{f : f \text{ is a function } \mathbb{N}_0 \to \mathbb{R}\}$.

Define $(f+g)(x) = f(x) + g(x)$ and $(cf)(x) = c\,f(x)$

Then V is a vector space under these operations.

↳ Very GOOD exercise to verify this...

Prop: Let V be a vector space.

(0) $0 \cdot v = 0_V$ for all $v \in V$.

(1) $-1 \cdot v$ is the additive inverse of v for all $v \in V$.

(2) $c \cdot 0_V = 0_V$

pf: Let V be a vector space and let $v \in V$ be arbitrary.

(0) $$0 \cdot v = (0+0) \cdot v = (0 \cdot v) + (0 \cdot v)$$

Hence, letting w denote the additive inverse of $0 \cdot v$

we have $$(0 \cdot v) + ((0 \cdot v) + w) = 0 \cdot v + 0_V = 0 \cdot v$$

while $$((0 \cdot v) + (0 \cdot v)) + w = 0 \cdot v + w = 0_V$$

Hence we have $0 \cdot v = 0_V$ as desired.

Rest of proof is next time... $\boxtimes$